Stat 513
Fall 2025
Problem Set 7
Topic 7: Parametric Families of Distributions

Due Sunday, December 14 at 23:59

Problem 1. Identifiability

Let $X \sim \text{Bernoulli}(p)$, $Y_1 \sim N(\mu_1, \sigma^2)$, $Y_2 \sim N(\mu_2, \sigma^2)$, and

$$Y = XY_1 + (1 - X)Y_2$$

where $[p, \mu_1, \mu_2, \sigma] \in (0, 1) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$. Consider the parametric family describing the distribution of Y with this parameter space. Is this parametric family identifiable?

Problem 2. Exponential Families

Show that the following distributions form exponential families with the specified parameters. For the first two, additionally use the cumulant function obtained from the exponential family form to derive the mean and variance of the distribution.

a) $\text{Binomial}(n, p)$ with $p \in (0, 1)$ and *fixed* n.

b) $\text{Poisson}(\lambda)$ with $\lambda > 0$.

c) $\text{Beta}(\alpha, \beta)$ Distribution with $\alpha > 0$ and $\beta > 0$, which is defined as an absolutely continuous distribution with the following pdf:

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1 - x)^{\beta-1}.$$

where $\Gamma(\cdot)$ is the Gamma function.

Problem 3. Characterization in Exponential Familes

Let X_1 and X_2 denote absolutely continuous one-dimensional random variables with density functions

$$f_1(x; \eta) = \exp(\eta x - k_1(\eta))\, h_1(x)$$

$$f_2(x; \eta) = \exp(\eta x - k_2(\eta))\, h_2(x)$$

respectively.

a) Show that if $k_1 = k_2$ for all η, then $X_1 \sim X_2$.

b) Show that if $\mathbb{E}[X_1; \eta] = \mathbb{E}[X_2; \eta]$ for all η, then $X_1 \sim X_2$.

c) Show that if $\mathbb{V}[X_1; \eta] = \mathbb{V}[X_2; \eta]$ for all η, then X_1 does not ncessarily have the same distribtuion as X_2.

Problem 4. Transforming Distributions

Show that

a) If $X \sim \text{Exponential}(\lambda)$, then $aX \sim \text{Exponential}(\lambda/a)$ for $a > 0$.

b) If $X \sim N(0, 1)$, then $X^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$.

c) If $X \sim N(0, 1)$ and $Y \sim N(0, 1)$, then $X^2 + Y^2 \sim \text{Exponential}\left(\frac{1}{2}\right)$.

d) If X denotes a Cauchy distributed random variable, then $X^2 + \frac{1}{X^2} \sim 4X^2 + 2$.

Problem 5. Order Statistics

Let $X_1, \ldots, X_n$ be i.i.d. from some distribution. The k-th *order statistic* is defined as defined as the k-th random variable when arranged in a sorted order, i.e. $X_{(1)} \leq X_{(2)} \leq \ldots, X_{(n)}$, where $X_{(1)}$ is the minimum, $X_{(n)}$ is the maximum, and the k-th order statistic is $X_{(k)}$.

a) Say that $X_1, \ldots, X_n \sim \text{Uniform}(0, 1)$. Show that $X_{(k)} \sim \text{Beta}(k, n + 1 - k)$.

b) In the same set-up as (a), show that as $n \to \infty$, the first order statistic (i.e., a minimum) converges to an Exponential(n) distribution (formally, $nX_{(1)} \xrightarrow{\mathcal{D}} \text{Exponential}(1)$).

Problem 6. Multivariate Normal

Let $X \in \mathbb{R}^n$ denote a random vector with an n-dimensional multivariate normal distribution with mean zero and covariance matrix Σ. Let $A \in \mathbb{R}^{n \times n}$ denote a real-valued matrix and consider the random vector $Y = AX$. What is the distribution of Y?

Bonus Problem. Pareto Distribution and Conditional Expectation

Let X denote an absolutely continuous random variable with density function

$$f(x) = \frac{\alpha}{x^{\alpha+1}} \text{ for } x > 1.$$

This is referred to as the Pareto distribution with parameter $\alpha > 0$. Let $Z = XY$ and $W = X/Y$. What are the values of $E[X|Z]$ and $E[W|Z]$?