Stat 513
Fall 2025
Problem Set 6
Topic 6: Moment Generating Functions, Characteristic Functions, and Cumulants

Due Wednesday, November 19 at 23:59

Problem 1. Positivity of the Laplace Transform.

Show that

$$(-1)^n \frac{\partial^n}{\partial t^n} L(t) \geq 0$$

for all $n \in \mathbb{N}$.

Problem 1 Solution

Consider $n = 1$. We can see that

$$\begin{aligned} \frac{\partial}{\partial t} L(t) &= \frac{\partial}{\partial t} \mathbb{E}\left[\exp(-tX)\right] \\ &= \frac{\partial}{\partial t} \int_0^\infty \exp(-tx) dF(x) \\ &= \int_0^\infty \frac{\partial}{\partial t} \exp(-tx) dF(x) \\ &= \int_0^\infty -t \exp(-tx) dF(x) \\ &= -\mathbb{E}\left[t \exp(-tX)\right]. \end{aligned}$$

Continuing this process, we can show that

$$\frac{\partial^n}{\partial^n t} L(t) = (-1)^n \mathbb{E}\left[t^n \exp(-tX)\right].$$

Dividing by $(-1)^n$ gives us

$$(-1)^n \frac{\partial^n}{\partial^n t} L(t) = \mathbb{E}\left[t^n \exp(-tX)\right] \geq 0$$

where the inequality comes from the fact that $t \geq 0$ and $\exp(tx) \geq 0$, so the expectation is non-negative for all n.

Problem 2. Generating Moments.

Complete the proof that if for a random variable X the moment generating function $M_X(t)$ exists for $t \in (-\delta, \delta)$ where $\delta > 0$, then

$$\mathbb{E}[X^j] = M_X^{(j)}(0)$$

for all $j = 1, 2, \ldots$.

Problem 2 Solution

We can show the statement

$$M_X^{(j)} = \mathbb{E}[X^j] + \sum_{k=1}^{\infty} \frac{t^k \mathbb{E}[X^{j+k}]}{k!}.$$

by induction, by taking the derivative of the above with respect to t to get

$$M_X^{(j+1)} = \sum_{k=1}^{\infty} \frac{t^{k-1} \mathbb{E}[X^k]}{(k-1)!}.$$

The result comes from taking out the first term of the sum and shifting the indices.

Problem 3. Deriving Characteristic Functions Derive characteristic functions for the following distributions:

a) X is a discrete random variable with

$$P(X = k) = \theta(1-\theta)^k$$

where $\theta \in (0, 1)$ and $k = 0, 1, \ldots$.

b) X is a continuous random variable with the following pdf:

$$f_X(x) = \frac{1}{2} \exp(-|x|)$$

for $x \in \mathbb{R}$. This is the *Laplace distribution*.

Problem 3 Solution

a) $\varphi(t) = \frac{\theta}{1-(1-\theta)e^{it}}$

b) $\varphi(t) = \frac{1}{1+t^2}$

Problem 4. Continuity of characteristic functions.

Show that characteristic functions are continuous with respect to t.

Problem 4 Solution We want to show that $\varphi(t+\epsilon) \to \varphi(t)$ as $\epsilon \to 0$, which can be accomplished by showing that $|\varphi(t+\epsilon) - \varphi(t)|$ goes to zero. We can bound using the absolute value at first:

$$|\varphi(t+\epsilon) - \varphi(t)| \leq \int_{-\infty}^{\infty} |\exp(ix(t+\epsilon)) - \exp(itx)|\, dF.$$

Since the function inside of the integral on the right hand-side is bounded, we can use the dominated convergence theorem to take the limit as $\epsilon \to 0$:

$$\lim_{\epsilon\to 0} |\varphi(t+\epsilon) - \varphi(t)| \leq \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} |\exp(ix(t+\epsilon)) - \exp(itx)|\, dF = \int_{-\infty}^{\infty} 0 dF = 0.$$

Problem 5. Poisson Distribution.

Recall that a random variable X is Poisson distributed with parameter $\lambda > 0$ if it is discrete and

$$P(X = k) = \frac{\lambda^k \exp(-\lambda)}{k!}$$

for all $k = 0, 1, \ldots$.

a) Derive the moment generating function of the Poisson distribution.

b) Show that each of the cumulants of the Poisson distribution are equal to λ.

Problem 5 Solution

a) The moment generating function is

$$\begin{aligned} M_X(t) &= \mathbb{E}\left[\exp(tX)\right] \\ &= \sum_{k=0}^{\infty} \exp(tk) \frac{\exp(-\lambda)\lambda^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\exp\left(tk - \lambda - k\log(\lambda)\right)}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\left(\lambda e^t\right)^k}{k!} \\ &= e^{-\lambda} \exp\left(\lambda e^t\right) \\ &= \exp\left(\lambda(e^t - 1)\right). \end{aligned}$$

where we use the power series expansion of e^x in reverse in the third step.

b) The cumulant generating function is

$$K(t) = \lambda \left(e^t - 1\right).$$

All derivatives of $K(t)$ are λe^t, so evaluating at $t = 0$ gives us that all cumulants are equal to λ.

Problem 6. Log-normal moment generating function.

A random variable X has a *log-normal* distribution if it is absolutely continuous with p.d.f.

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log x - \mu)^2}{2\sigma^2}\right).$$

Show that the moment generating function of the log-normal distribution does not exist (in a neighborhood around zero), even though $\mathbb{E}[X^n]$ exists for all $n = 0, 1, \ldots$.

Problem 6 Solution For simplicity consider the case that $\mu = 0$ and $\sigma = 1$. Letting $y = \log(x)$, then $dy = \frac{1}{x}dx$ and we get

$$\begin{aligned}
\mathbb{E}\left[X^n\right] &= \int_0^\infty \frac{1}{\sqrt{2\pi}} x^{n-1} \exp\left(-\frac{(\log x)^2}{2}\right) dx \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi}} x^n \exp\left(-\frac{y^2}{2}\right) dy. \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(ny\right) \exp\left(-\frac{y^2}{2}\right) dy.
\end{aligned}$$

This is the moment generating function of a standard normal distribution evaluated at $t = n$, so

$$\mathbb{E}\left[X^n\right] = \exp\left(n^2/2\right).$$

However, the moment generating function

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{x} \exp\left(tx - \frac{(\log x)^2}{2}\right) dx$$

diverges when $t \geq 0$, since $\exp(tx)$ goes to infinity as $x \to \infty$; as such, the moment generating function does not exist in a neighrbohood around zero.

Problem 7. Kurtosis.

Recall that Kurtosis of a random variable X is defined as

$$\frac{K_4(X)}{\text{Var}[X]^2}$$

where $K_4(X)$ denotes the fourth cumulant of X, which has the following expression:

$$K_4(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^4\right] - 3\text{Var}[X]^2.$$

a) Show that for any random variable X,

$$\text{Kurtosis}(X) \geq -2.$$

b) Let $X \sim \text{Bernoulli}(p)$ for $p \in (0, 1)$. Derive an expression for the Kurtosis of X, and show that it achieves its minimum value of $\text{Kurtosis}(X) = -2$ at $p = \frac{1}{2}$.

Problem 7 Solution

a) We want to show that

$$\mathbb{E}\left[(X - \mathbb{E}[X])^4\right] - 3\text{Var}(X)^2 \geq -2\text{Var}(X)^2.$$

This is equivalent to

$$\mathbb{E}\left[(X - \mathbb{E}[X])^4\right] \geq \text{Var}(X)^2.$$

Let $Y = (X - \mathbb{E}[X])^2$. Then we can write the above expression as

$$\mathbb{E}[Y^2] \geq \mathbb{E}[Y]^2.$$

This holds via Jensen's inequality, and the fact that $g(x) = x^2$ is a convex function.

b) We can see that

$$\mathbb{E}\left[(X - \mathbb{E}[X])^4\right] = p(1-p)\left((1-p)^3 + p^3\right).$$

Subtracting $3\text{Var}[X]^2 = 3p^2(1-p)^2$ we get

$$\begin{aligned} &p(1-p)\left((1-p)^3 + p^3\right) - 3p^2(1-p)^2 \\ =\ &p(1-p)\left(1 - 3p + 3p^2 - p^2 + p^3 - 3p + 3p^2\right) \\ =\ &p(1-p)\left(1 - 6p + 6p^2\right). \end{aligned}$$

Dividing by $p^2(1-p)^2$, we get

$$\text{Kurtosis}(X) = \frac{1 - 6p(1-p)}{p(1-p)}.$$

Plugging in $p = \frac{1}{2}$ we can see that $\text{Kurtosis}(X) = -2$. We can see this is the minimum value for all p since setting $f(p) = \frac{1-6p(1-p)}{p(1-p)}$ we can see that

$$\frac{\partial}{\partial p} f(p) = \frac{1 - 2p}{(p - p^2)} = 0$$

which means that $f(p)$ achieves its minimum at $p = \frac{1}{2}$.

Problem 8. Random Sum.

Let $J \sim \text{Poisson}(\lambda)$, let $X_1, X_2, \ldots$ be independent and identically distributed random variables with mean μ and variance σ^2, and let

$$S = \sum_{j=0}^{J} X_j.$$

Assume that X_j have a moment generating function $M_X(t)$ which exists within a neighborhood of zero. Derive an expression for the moment generating function of S (hint: break up the expectation into an inner expectation conditioned on the value of J, and use indicator functions).

Problem 8 Solution We want to evaluate

$$M_S(t) = \mathbb{E}\left[\exp(tS)\right].$$

We will write S as $\sum_{n=1}^{\infty} X_j \mathbb{1}\{j \leq J\}$ which will make things easier. Using our expectation rules, we can see that

$$\begin{aligned}
M_S(t) &= \mathbb{E}\left[\mathbb{E}\left\{\exp\left(\sum_{n=1}^{\infty} X_j \mathbb{1}\{j \leq J\}\right) \middle| J\right\}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left\{\prod_{j=1}^{\infty} \exp\left(X_j \mathbb{1}\{j \leq J\}\right) \middle| J\right\}\right] \\
&= \mathbb{E}\left[\prod_{j=1}^{\infty} \mathbb{E}\left[\exp\left(X_j\right)^{\mathbb{1}\{j \leq J\}} \mid J\right]\right] \\
&= \mathbb{E}\left[\prod_{j=1}^{\infty} M_{X_j}(t)^{\mathbb{1}\{j \leq J\}}\right] \\
&= \mathbb{E}\left[M_X(t)^{\sum_{j=1}^{J} \mathbb{1}\{j \leq J\}}\right] \\
&= \mathbb{E}\left[M_X(t)^J\right] \\
&= \mathbb{E}\left[\exp\left(J \log(M_X(t))\right)\right] \\
&= M_J\left(\log(M_X(t))\right) \\
&= \exp\left(\lambda\left(M_X(t) - 1\right)\right)
\end{aligned}$$

using the moment generating function of the Poisson distribution for the last step.