Stat 513

Fall 2025

Problem Set 5

Topic 5: Law of Large Numbers and Types of Convergence

Due Wednesday, November 12 at 23:59

Problem 1. A Case where Convergence in Distribution Implies Convergence in Probability

Let X denote a random variable such that $P(X = a) = 1$ for some constant a, and let X_n denote a sequence of random variables on the same probability space. Show that

$$X \xrightarrow{\mathcal{D}} a \text{ implies } X \xrightarrow{\mathcal{P}} a.$$

Problem 1 Solution Say that $X \xrightarrow{\mathcal{D}} a$. Then

$$\lim_{n\to\infty} F_n(x) = \mathbb{1}\{x \geq c\}.$$

We want to show that the probability that $|X_n - a| \geq \epsilon$ goes to zero as $n \to \infty$. In order to write this probability in terms of F_n, we can split it up:

$$\begin{aligned} P\left(|X_n - a| \geq \epsilon\right) &= P\left(X_n \leq a - \epsilon\right) + P\left(X_n > a + \epsilon\right) \\ &\leq F_n(a - \epsilon) + 1 - F_n(a + \epsilon). \end{aligned}$$

Taking the limit on both sides as $n \to \infty$, we get

$$P\left(|X_n - a| \geq \epsilon\right) \leq 0 + 1 - 1 = 0$$

which implies that $X \xrightarrow{\mathcal{P}} a$.

Problem 2. Convergence Almost Surely Implies Convergence in Distribution

Show directly (i.e., without using the fact that almost sure convergence implies convergence in probability, which implies convergence in distribution) that

$$X_n \xrightarrow{a.s} X \text{ implies } X_n \xrightarrow{\mathcal{D}} X.$$

Problem 2 Solution There are multiple ways to do this. In one way, we can use the fact that if

$$\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)]$$

for all bounded, continuous functions f, then $X_n \xrightarrow{\mathcal{D}} X$. Note that if $X_n(\omega) \to X(\omega)$, then $f(X_n(\omega)) \to f(X(\omega))$ because f is continuous. Since "almost surely" means that this holds with probability one, it follows that $\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)]$, as required.

Problem 3. Convergence of Borel Sets

Let B denote a Borel set in $\mathbb{R}$. Is it true that $X_n \xrightarrow{\mathcal{D}} X$ implies

$$\lim_{n\to\infty} P(X_n \in B) = P(X \in B)?$$

Problem 3 Solution

No, consider $X_n \sim \text{Uniform}(-1/n, 0)$. Let $X \equiv 1$, then can see that $X_n \xrightarrow{\mathcal{D}} 1$. Consider $B = (-1, 0)$. Then, $P(X_n \in B) = 1$ for all n, so $\lim_{n\to\infty} P(X_n \in B) = 1$, however $P(X \in B) = 0$.

Problem 4. Example of Convergence in Distribution

Let $\lambda > 0$, and let $Y \sim \text{Poisson}(\lambda)$, meaning that Y is a discrete random variable such that

$$P(Y = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{for} \quad k \in \mathbb{N}_0.$$

Let $X_n \sim \text{Binomial}(n, p_n)$, meaning that X_n is a discrete random variable such that

$$P(X_n = k) = \binom{n}{k} p_n^k (1 - p_n)^{n-k} \quad \text{for} \quad 0 \le k \le n.$$

Show that if $p_n = \lambda/n$, then $X_n \xrightarrow{\mathcal{D}} Y$.

Problem 4 Solution

We can write the p.m.f. of the X_n as

$$\lim_{n\to\infty} P(X_n = k) = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(\frac{n-\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n!(n-\lambda)^{n-k}}{n(n-k)!} = \frac{\lambda^k}{k!} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} e^{\lambda}$$

which is the Poisson probability mass function, as required.

Problem 5. Lack of Convergence

Show an example of a sequence of random variables that does not converge in distribution to any random variable.

Problem 5 Solution Let $X_n \sim \text{Uniform}(0, 1/n)$. Then $F_n(x)$ will converge to $F(x) = \mathbb{1}\{x \leq 0\}$, which is not right-continuous and as such is not a valid distribution function.

Problem 6. Weak Law of Large Numbers with Dependence

Consider a sequence of random variables denoted by X_n, with a common mean $\mathbb{E}[X_i] = \mu$ and unit variance $V[X_i] = 1$. Instead of assuming that X_n are independent, let the covariance between X_i and X_j be defined as a function of the distance between their indices; specifically,

$$\text{Cov}\,(X_i, X_j) = f(i - j)$$

for some function $f : \mathbb{N} \to \mathbb{R}$, where we recall that $\text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. Show that if $\lim_{i\to\infty} f(i) = 0$, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\mathcal{P}} \mu.$$

Problem 6 Solution Using Chebyshev's inequality, it suffices to show that

$$\text{V}\left[\overline{X}_n\right] = \frac{1}{n^2}\text{V}[\overline{X}_n] \to 0$$

where $\overline{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$. We can start with the identity

$$V\left[\sum_{j=1}^{n} X_j\right] = \sum_{n=1}^{\infty} \text{V}[X_j] + \sum_{j>i} \text{Cov}[X_i, X_j].$$

The first term will go to zero for reasons analogous to the "regular" weak law of large numbers, so let's look at the second term. We can write this term as

$$\sum_{j>i} \text{Cov}[X_i, X_j] = \sum_{j=1}^{n} (n - j) f(j)$$

using the fact that for each lag of size j, there will be $n - j$ pairs that are j indices apart. Fix $\epsilon > 0$, and since $\lim_{i\to\infty} f(i) = 0$ we can let N be large enough such that $f(k) \leq \epsilon$ for all $k \geq N$. Taking n to be greater than N, we can write this sum as

$$\begin{aligned}\sum_{j=1}^{n} (n - j) f(j) &= \sum_{j=1}^{N} (n - j) f(j) + \sum_{j=N+1}^{n} (n - j) f(j). \\ &\leq \sum_{j=1}^{N} (n - j) f(j) + \epsilon(n - N)^2\end{aligned}$$

Where the inequality comes from the fact that $f(j) \leq \epsilon$, $(n-j) \leq (n-N)$, and the fact that the second summation has $n-N$ terms. Now, we can add back the $\frac{1}{n^2}$ term to see that

$$\begin{aligned} &\frac{1}{n^2}\left(\sum_{j=1}^{N}(n-j)f(j) + \epsilon(n-N)^2\right) \\ =&\frac{1}{n^2}\sum_{j=1}^{N}(n-j)f(j) + \epsilon\left(1-\frac{N}{n}\right)^2 \end{aligned}$$

which will go to zero as $n \to 0$ since N is fixed and first sum is a constant and $\frac{N}{n} \to 0$. Since ϵ is arbitrary, $\mathrm{V}[\overline{X}_n] \to 0$ as required.

Bonus Problem. Convergence in Expectation

Let a denote a constant and X_n a sequence of random variables. Show by example that $X_n \xrightarrow{a.s} c$ does not necessarily imply that $\lim_{n\to\infty} \mathbb{E}[X_n] = c$.

Problem 6 Solution Let $X_n \sim \text{Bernoulli}(1/n^2)$, and consider $Y_n = n^2 X_n$. Then,

$$\mathbb{E}[Y_n] = n^2 \frac{1}{n^2} = 1$$

so $\lim_{n\to\infty} \mathbb{E}[Y_n] = 1$. Let

$$A_n = \{\omega : Y_j(\omega) = 0 \text{ for all } j \geq n\}.$$

and let $B_n = A_n^c$. Note that the sets B_n form an increasing sequence, so we have

$$\lim_n P(B_n) = P(\lim_n B_n) = P\left(\bigcup_{j=1}^{\infty} B_j\right)$$

. We can see that

$$P(B_n) \leq P\left(\bigcup_{j=n}^{\infty}\{Y_j(\omega) \neq 0\}\right) \leq \sum_{j=n}^{\infty} P(\{\omega : Y_j(\omega) = 0\}) = \sum_{j=n}^{\infty}\frac{1}{n^2}.$$

Taking the limit, we see that $\lim_n P(B_n) = 0$, which means that $P\left(\bigcup_{j=1}^{\infty} B_j\right) = 0$. Finally, we can see that

$$P\left(\left(\bigcup_{j=1}^{\infty} B_j\right)^c\right) = P\left(\{\omega : Y_n(\omega) \to 0\}\right) = 1$$

which means that $Y_n \xrightarrow{a.s} 0$.