Stat 513

Fall 2025

Problem Set 5

Topic 5: Law of Large Numbers and Types of Convergence

Due Wednesday, November 12 at 23:59

Problem 1. A Case where Convergence in Distribution Implies Convergence in Probability

Let X denote a random variable such that $P(X = a) = 1$ for some constant a, and let X_n denote a sequence of random variables on the same probability space. Show that

$$X \xrightarrow{\mathcal{D}} a \text{ implies } X \xrightarrow{\mathcal{P}} a.$$

Problem 2. Convergence Almost Surely Implies Convergence in Distribution

Show directly (i.e., without using the fact that almost sure convergence implies convergence in probability, which implies convergence in distribution) that

$$X_n \xrightarrow{a.s} X \text{ implies } X_n \xrightarrow{\mathcal{D}} X.$$

Problem 3. Convergence of Borel Sets

Let B denote a Borel set in $\mathbb{R}$. Is it true that $X_n \xrightarrow{\mathcal{D}} X$ implies

$$\lim_{n\to\infty} P(X_n \in B) = P(X \in B)?$$

Problem 4. Example of Convergence in Distribution

Let $\lambda > 0$, and let $Y \sim \text{Poisson}(\lambda)$, meaning that Y is a discrete random variable such that

$$P(Y = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{for} \quad k \in \mathbb{N}_0.$$

Let $X_n \sim \text{Binomial}(n, p_n)$, meaning that X_n is a discrete random variable such that

$$P(X_n = k) = \binom{n}{k} p_n^k (1 - p_n)^{n-k} \quad \text{for} \quad 0 \leq k \leq n.$$

Show that if $p_n = \lambda/n$, then $X_n \xrightarrow{\mathcal{D}} Y$.

Problem 5. Lack of Convergence

Show an example of a sequence of random variables that does not converge in distribution to any random variable.

Problem 6. Weak Law of Large Numbers with Dependence

Consider a sequence of random variables denoted by X_n, with a common mean $\mathbb{E}[X_i] = \mu$ and unit variance $V[X_i] = 1$. Instead of assuming that X_n are independent, let the covariance between X_i and X_j be defined as a function of the distance between their indices; specifically,

$$\mathrm{Cov}\left(X_i, X_j\right) = f(i - j)$$

for some function $f : \mathbb{N} \to \mathbb{R}$, where we recall that $\mathrm{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. Show that if $\lim_{i\to\infty} f(i) = 0$, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\mathcal{P}} \mu.$$

Bonus Problem. Convergence in Expectation

Let a denote a constant and X_n a sequence of random variables. Show by example that $X_n \xrightarrow{a.s} c$ does not necessarily imply that $\lim_{n\to\infty} \mathbb{E}[X_n] = c$.