Stat 513

Fall 2025

Problem Set 4

Topic 4: Conditional Distributions and Conditional Expectation

Due Sunday, October 19 at 23:59

Problem 1. Tightness of Chebyshev's Inequality

Consider a random variable X with range $\{-a, 1, a\}$ for some $a > 0$ such that:

$$P(X = -a) = p$$
$$P(X = 0) = 1 - 2p$$
$$P(X = a) = p.$$

Show that for some value of k, Chebyshev's inequality holds with equality.

Problem 2. Cantelli's inequality

i) Prove the following inequality:

$$P(X - \mathbb{E}[X] \geq \lambda) \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}$$

where $\lambda \geq 0$.

ii) When is Cantelli's inequality better than Chebyshev's inequality?

Problem 3. A Sum Rule for Expectations

Show that if the range of X is the natural numbers, then

$$E[X] = \sum_{n=1}^{\infty} P(X \geq n).$$

Problem 4. Conditional Densities for Absolutely Continuous Distributions

Let X and Y denote real-valued random variables such that the distribution of (X, Y) is absolutely continuous with density function

$$p(x, y) = \frac{1}{x^3 y^2}, \quad x > 1,\ y > 1/x.$$

Find conditional distributions for X given $Y = y$ and Y given $X = x$.

Problem 5. Mixed Distribution Let (X, Y) denote a two-dimensional random vector with range $(0, \infty) \times \{1, 2\}$ such that for any set $A \subset (0, \infty)$ and $y \in \{1, 2\}$,

$$P(X \in A, Y = y) = \frac{1}{2} \int_A y \exp(-yx) dx.$$

Find conditional distributions for X given $Y = y$ and for Y given $X = x$.

Problem 6. Non-uniqueness of Conditional Probabilities Using the joint distribution from the previous problem, describe two conditional distributions (i.e., set functions $q_1(\cdot, y)$ and $q_2(\cdot, y)$ that satisfy the definition of conditional probability) that differ on an uncountable set.

Problem 7. Conditional Distributions as a Limit

Let X and Y denote real-valued random variables such that the distribution of (X, Y) is absolutely continuous with density function f, and let f_X denote the marginal density function of X. Suppose that there exists a point x_0 such that $f_X(x_0) > 0$, f_X is continuous at x_0, and for almost all y, $f(\cdot, y)$ is continuous at x_0. Let $A \subset \mathbb{R}$. For each $\epsilon > 0$, let

$$d(\epsilon) = P(Y \in A | x_0 \leq X \leq x_0 + \epsilon).$$

Show that

$$P(Y \in A | X = x_0) = \lim_{\epsilon \to 0} d(\epsilon).$$

Problem 8. Sums of Conditional Expectations

Let X denote a real valued random variable with range $\mathcal{X}$, such that $E[|X|] < \infty$. Let $A_1, \ldots, A_n$ denote disjoint subsets of $\mathcal{X}$. Show that

$$E(X) = \sum_{i=1}^{N} E[X | X \in A_j] P(X \in A_j).$$

Bonus Problem. **Conditional Expectation as Projection** Say that Z is a random variable such that $E[X^2] < \infty$. Say that there exists a random variable W such that $E[(Y - W)h(Z)] = 0$ for all random variables $h(Z)$ such that $E[h(Z)^2] < \infty$. Such a random variable may be referred to as the projection of Y

onto the subspace of functions of Z, or $W \equiv P_Z Y$. Show that

$$P_Z Y = E[Y|Z].$$