Stat 513
Fall 2025
Problem Set 2
Topic 2: Random Variables, Distribution Functions, and Continuity

Total: 50 points

Due Sunday, September 21 at 23:59

1 Continuity

Problem 1. Continuity

a) Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}((a,b))$ is open for any open interval $(a,b) \subset \mathbb{R}$.

b) Show that for a continuous function $f : \mathbb{R} \to \mathbb{R}$, for any fixed u the set

$$A = \{x \in \mathbb{R} : f(x) = u\}$$

is closed.

Problem 2. Limits of continuous functions

Consider continuous functions $f_1, f_2, \ldots$ on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be function defined as $f(x) = \lim_{n\to\infty} f_n(x)$ for all x. Show by example that f is not necessarily continuous. Conclude that a sequences of continuous random variables may converge in distribution to a discrete random variable (i.e., their distribution functions converge to a distribution function of a discrete random variable).

2 Distribution Functions

Problem 3. Properties of Distribution Functions

Let X denote a random variable on $(\Omega, \mathcal{F}, P)$, with distribution function F. Define the notation

$$F(x-) = \lim_{h\downarrow 0} F(x-h)$$

and

$$F(x+) = \lim_{h\downarrow 0} F(x+h)$$

to denote the "limit from below/left" and the "limit from above/right".

a) Using a variation of the proof shown in class that $\lim_{x\to\infty} F(x) = 1$, show that

$$\lim_{x\to-\infty} F(x) = 0.$$

b) Show that

$$F(x) = F(x-) + P(X = x).$$

c) Conclude that F is continuous if and only if $P(X = x) = 0$ for all $x \in \mathbb{R}$.

Problem 4. Quantile Functions

Let X denote a random variable on probability space $(\Omega, \mathcal{F}, P)$ with distribution function F, define the *quantile function* $Q : (0,1) \to \mathbb{R}$ as

$$Q(u) = \inf\{x : F(x) \geq u\}.$$

a) Show that

$$P(X \leq Q(u)) \geq u.$$

This explains the naming of Q as the "quantile function".

b) What is the quantile function for $X \sim \text{Bernoulli}(1/2)$?

c) Show that the quantile function is monotonic non-decreasing.

d) Show that Q is a *psuedo-inverse* of F, in the sense that

$$Q(F(Q(u))) = Q(u) \text{ for all } u \in (0,1)$$

and

$$F(Q(F(x))) = F(x) \text{ for all } x \in \mathbb{R}.$$

It may help to show that $Q(F(x)) \leq x$ for all $x \in \mathbb{R}$.

e) Show that if F^{-1} is a function (in the sense that $F^{-1}(u)$ is a singleton element for all u), then $F^{-1} = Q$.

3 Random Variables

Problem 5. Equivalent Distributions

a) Let X_1 and X_2 be random variables on probability space $(\Omega, \mathcal{F}, P)$ such that $X_1 \sim X_2$. Let $g : \mathbb{R} \to \mathbb{R}$ denote a function such that for borel sets B, $g^{-1}(B)$ is borel. Show that $g(X_1) \sim g(X_2)$.

b) Let X_1 and X_2 be random variables defined on $(\Omega, \mathcal{F}, P)$ with distribution functions F_1 and F_2. Show if that for any interval $(a, b) \subset \mathbb{R}$,

$$F_1(b) - F_1(a) = F_2(b) - F_2(a),$$

then $X_1 \sim X_2$. You can use the unique characterization result asserted in class.

Problem 6. Sequences and Sums of Random Variables

Consider the probability space $(\Omega, \mathcal{F}, P)$ for $\Omega = (0, 1)$, Borel sets $\mathcal{F}$ and uniform probability measure P. As before, define $X_i = b_i(\omega)$ where b_i refers to the i-th digit in the binary representation of ω.

a) Show that if X_i is a random variable for all i, then

$$S(\omega) = \sum_{i=1}^{n} X_i(\omega)$$

is a random variable for any value of n.

b) **Changed to be an extra bonus problem.**

Problem 7. Inequalities of Random Variables

Let X_i be defined as in Problem 6.

a) Let $Z_1 = X_1 + X_2$ and $Z_2 = X_2 + X_3$. Determine the value of $P(Z_1 > Z_2)$ by explicitly defining the subset of $(0, 1)$ corresponding to this event, and then taking its length.

b) Calculate the same probability as in the previous part by defining the random variable $Y = Z_1 - Z_2$, and using the fact that the X_i are independent.

Problem 8. Independence of Random Variables Let X and Y denote two random variables defined on $(\Omega, \mathcal{F}, P)$.

a) Find non-constant functions g and h such that $g(X)$ and $h(Y)$ are independent, even when X and Y are not independent.

b) Show that if X and Y are independent, then $g(X)$ and $h(Y)$ are independent for any pair of functions g and h.

Bonus Problem. Limits of Quantile Functions

1) Let $F_1, F_2, \ldots, F_n, \ldots$, and F be distribution functions with corresponding quantile distribution functions $Q_1, Q_2, \ldots$, and Q. Suppose that

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \text{for all continuity points } x \text{ of } F.$$

Show that

$$Q(u) \le \liminf_n Q_n(u) \le \limsup_n Q_n(u) \le \lim_{h\to 0} Q(u+h)$$

and conclude that

$$\lim_{n\to\infty} Q(u) = Q(u) \quad \text{for all continuity points } u \text{ of } Q.$$

2) (From problem 6, note change from the previous version): Show that if $X_i(\omega) = b_i(\omega)$ are random variables with respect to the same probability space $(\Omega, \mathcal{F}, P)$, then

$$\left\{\omega : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_n(\omega) = a\right\} \in \mathcal{F}$$

for any $a \in (0,1)$.