Stat 513
Assignment 1
Topic 1: Probability and Set Theory
Fall 2025

Total: 50 points

Due Sunday, September 7 at 23:59

1 Set Theory

Problem 1. Countability..

The *algebraic numbers* are defined as the set of roots of polynomials with integer coefficients. Formally,

$$A = \left\{ x : \exists N, a_0, a_1, ...a_N \in \mathbb{Z} \text{ s.t. } \sum_{i=0}^{N} a_i x^i = 0 \right\}.$$

Is A countable or uncountable? Show your answer by either demonstrating the existence of a bijection, or showing that no such bijection could exist.

Problem 1 Solution

Consider the set

$$B = \bigcup_{N=1}^{\infty} \{(a_0, ..., a_N) : a_0, ..., a_N \in \mathbb{Z}\}.$$

Note that B is "larger" than A, in the sense that we can, for example, let $g : A \to B$ to be such that $g(x)$ corresponds to one of the possible polynomials that has x as a root (for example, by taking the smallest N, then the smallest a_0, ...). Then, g is a bijection with its $g(A) \subset B$; as such, it is sufficient to prove that B is countable.

Since, as asserted in class, the countable union of countable sets is countable, it is sufficient to show that component sets of the union in the definition of B are each countable. Since each of these can be seen as equivalent to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \cdots$, each of these can be countable using the (informal) "diagonal" argument used to show that the rational numbers $\mathbb{Q}$ are countable.

For a more explicit example, fix N and consider $y = (a_0, ..., a_N)$. Since $\mathbb{Z}$ is countable, we can take a_0 to be equivalent to its corresponding natural number. Then if N is even, let $f(y)$ correspond to a binary decimal expansion with N zeros, a_0 ones, a_1 zeros, ... a_N ones, and then infinite zeros; if N is odd start with ones, so that you "end" with ones. Then, you can reconstruct $(a_0, ..., a_N)$ by first "reading off" the value of N, and then repeatedly looking at the number of zeros/ones until you've determined the value of a_N. Since the rest of the digits are zero, this is a rational number, and the rational numbers are countable.

Problem 2. Countability and Density of Sets

As in class, let $b_i(x)$ denote the i-th binary digit of $x \in (0,1)$. The set of normal numbers between zero and one is defined as the following set:

$$A = \left\{ x \in (0,1) \middle| \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} b_i(x) = \frac{1}{2} \right\}.$$

i) Show that A is dense in $(0,1)$. *Hint:* For a given ϵ, look at the first n digits of the binary expansion for an appropriate value of n.

ii) Show that the complement of A is also dense in $(0,1)$.

Problem 2 Solution

(i). Let $\epsilon > 0$ and let m be large enough such that $\frac{1}{2^m} < \epsilon$. For $x \in (0,1)$, let $a \in A$ be such that

$$b_i(a) = \begin{cases} b_i(x) & i \leq m \\ 0 & i > m, \text{ i is even} \\ 1 & i > m, \text{ i is odd} \end{cases}$$

Then we can see that (taking $n > m$ inside of the limits) that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} b_i(x) &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{m} b_i(a) + \lim_{n\to\infty} \frac{1}{n} \sum_{i=m+1}^{n} b_i(a) \\ &= 0 + \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

So, we can see that a is a normal number, and also that

$$|a - x| = \sum_{i=m+1}^{\infty} \frac{|b_i(a) - b_i(x)|}{2^n} \leq \frac{1}{2^m} < \epsilon$$

(ii) The logic here is the same as the above, but you can let $b_i(x) = 0$, for example, for $i > m$.

Problem 3. Lim-sup and lim-inf of sets.

Consider a countable sequence of sets $A_1, A_2, \ldots$. The lim-sup and lim-inf of this sequence are defined as follows:

$$\liminf A_i = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$$

and

$$\limsup A_i = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i.$$

i) Let $A_i = \left[0, \frac{1}{i}\right]$ if i is odd and $A_i = [0,1]$ if i is even. What are $\limsup A_i$ and $\liminf A_i$?

ii) Show that if $A_i \subset A_{i+1}$ for all i, then the liminf and limsup of $\{A_i\}_{i=1}^{\infty}$ are equal to each other and to the (infinite) union. Show the analogous result if $A_i \supset A_{i+1}$ with respect to the infinite intersection.

iii) Show that

$$(\liminf A_i)^c = \limsup A_i^c$$

and

$$(\limsup A_i)^c = \liminf A_i^c.$$

Problem 3 Solution

i)

$$\limsup A_i = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i = \bigcap_{n=1}^{\infty} [0,1] = [0,1].$$

$$\liminf A_i = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i = \bigcup_{n=1}^{\infty} \{0\} = \{0\}.$$

ii) If $A_i \subset A_{i+1}$, then

$$\bigcup_{i=n}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i$$

$$\bigcap_{i=n}^{\infty} A_i = A_n.$$

So,

$$\limsup A_i = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i = \liminf A_i.$$

The other case of "decreasing" sets is similar.

iii) By DeMorgan's laws for infinite unions/intersections, we have

$$(\limsup A_i)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \right)^c = \bigcup_{n=1}^{\infty} \left(\bigcup_{i=n}^{\infty} A_i \right)^c = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i^c = \liminf A_i^c$$

and similar for $\liminf$.

Problem 4. The Cantor Set

Consider the set of sequences of elements of $\{0, 1, 2\}$:

$$\mathcal{T} = \left\{ \{x_i\}_{i=1}^{\infty} \middle| x \in \{0, 1, 2\} \right\}.$$

Similar to the set of binary sequences, we can define $t_i(x)$ as the i-th *ternary* digit of $x \in (0, 1)$, and establish a (psuedo) bijection with the unit interval:

$$f(\{x_i\}) = \sum_{i=1}^{\infty} \frac{t_i(x)}{3^i}.$$

(Note that this is not a true bijection as written because we would need to establish a condition for equivalent expansions similar to what we did for the binary digits, since $.022\ldots$ and $.100\ldots$ are both equal to $1/3$; however we will ignore this complication as justified by Problem 5(i)).

Given this definition, the *Cantor set* can be defined as

$$\mathcal{C} = \{x \in (0, 1) \big| t_i(x) \neq 1\}.$$

i) Show that the Cantor set as defined above is equivalent to defining collections of sets C_n for all $n \in \mathbb{N}$ through the following iterative process:

   a) Initialization: Let $C_0 = \{(0, 1)\}$.

   b) Construct C_{n+1} from C_n by removing the middle third of each of the intervals of C_n, i.e.

$$C_{n+1} = \left\{ \left[a, a + \frac{b-a}{3}\right], \left[a + \frac{2(b-a)}{3}, b\right] \middle| \forall [a, b] \in C_n \right\}.$$

   and taking the infinite intersection of the union of each of the C_n:

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \bigcup_{A \in C_n} A.$$

ii) Show that the Cantor set is closed (i.e., contains all of its limit points).

iii) A set S is *nowhere dense* in $\mathcal{X}$ if for all open subsets $E \subset \mathcal{X}$, S is not dense in E. Show that the Cantor set is nowhere dense in $(0, 1)$.

iv) Using the uniform probability space, show that $P(\mathcal{C}) = 0$ by showing that $P(\mathcal{C}) < \epsilon$ for all $\epsilon > 0$.

Problem 4 Solution

i) It is sufficient to show that at each level n,

$$\bigcup_{A \in C_n} A = \{x : t_i(x) \neq 1 \ \forall i \leq n\}.$$

We will show this by proving the following claim through induction: at each level n, C_n consists of the intervals

$$\left[x, x+\frac{1}{3^n}\right] \ \forall x \in A_n$$

where

$$A_n = \{x : t_i(x) \neq 1 \ \forall i \leq n \text{ and } t_i(x) = 0 \, \forall i > n\}.$$

This is true when $n = 0$ and $x = 0$ since the interval is $[0, 1]$.

Suppose by that is true for n. We can see that each element $x \in A_n$ corresponds to two elements in A_{n+1}, x and $x + \frac{2}{3^{n+1}}$. When we construct C_{n+1}, the construction will produce the two intervals

$$\left[x, x+\frac{1}{3^{n+1}}\right], \ \left[x+\frac{2}{3^{n+1}}, x+\frac{1}{3^n}\right].$$

We can see that the first interval corresponds to x in A_{n+1}. Re-writing the second interval as

$$\left[x+\frac{2}{3^{n+1}}, \left(x+\frac{2}{3^{n+1}}\right)+\frac{1}{3^{n+1}}\right]$$

we can see that this corresponds to $x + \frac{2}{3^{n+1}} \in A_{n+1}$. Since this holds for any element of A_n, we have shown the induction hypothesis for C_{n+1} and as such for all C_n. The result follows by taking the infinite intersection.

ii) Let x denote a limit point of the Cantor set. Let $\epsilon = \frac{1}{3^n}$. By the definition of a limit point there exists a $x_n \in \mathcal{C}$ such that $|x - x_n| < \epsilon$. This implies that $t_i(x) = t_i(x_n) \neq 1$ for all $i < n$. Since n is arbitrary, this means that $t_i(x) \neq 1$ for all i, which means that $x \in \mathcal{C}$, so since $\mathcal{C}$ contains all of its limit points and is as such closed.

iii) Let $A \subset (0, 1)$ denote an open set. By definition, for any x there exists an $\epsilon > 0$ such that $N_\epsilon(x) \subset A$. Denote the endpoints of this interval as (a, b). Since $a \neq b$, there must exist some i such that $t_i(a) \neq t_i(b)$; take i to be the first such value. Then let y be such that $t_j(y) = t_j(a) = t_j(b)$ for all $j < i$ but $t_i(y) = 1$. Then $a < y < b$ but y is not in $\mathcal{C}$, so $\mathcal{C}$ cannot be dense in A.

iv) Since

$$P(C_n) = \sum_{i=1}^{2^n} \frac{1}{3^n} = \left(\frac{2}{3}\right)^n$$

and $P(\mathcal{C}) < P(C_n)$ for all n, then take n to be large enough such that $\left(\frac{2}{3}\right)^n < \epsilon$ for any $\epsilon < 0$ to show that $P(\mathcal{C}) = 0$

2 Basic Probability

Problem 5. Binary Sequences

i) Show that, when represented in base-2, $.1000... = .0111... = \frac{1}{2}$.

ii) Taking $(\Omega, \mathcal{F}, P)$ to be $\Omega = (0,1)$, $\mathcal{F}$ as the Borel sets, and P as the uniform probability measure, show that

$$P\left(\left\{x \middle| \exists N \text{ s.t. } b_j(x) = 0 \ \forall j \geq N\right\}\right) = 0.$$

(In other words, the probability that a given binary sequence ends in all zeros is zero). This allows us to use our bijection between the "non-terminating" binary sequences and the interval $(0,1)$ without any loss of generality.

Problem 5 Solution

i) We can see that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{2^n} = 1$$

implies

$$\sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{2}.$$

ii) Because each element of this set is a rational number, it is countable, and as such must be of probability one as shown in class.

Problem 6. Infinite Sequences of Coin Flips

For the following parts, consider the event space of infinite sequences of zero-one coin flips:

$$\Omega = \left\{ \{x_i\}_{i=1}^{\infty} \middle| x_i \in \{0,1\} \right\}.$$

i) Using the $\sigma-$algebra generated by evenly sized intervals of width 1/8 (i.e., $\mathcal{F}$ composed of the sets $A_i = (i/8, (i+1)/8)$ along with union and complements) derive the probability of the second and third coin flips being heads.

ii) What is the smallest σ-algebra (i.e., composed of the fewest number of sets) that will allow you to evaluate the probability of the second and third coin flips being heads?

iii) What is the smallest σ-algebra (i.e., composed of the fewest number of sets) that will allow you to evaluate the probabilities that the second and third coin flips take *any* value? (i.e, (H,H), (T,T), (H,T), (T,T)).

Problem 6 Solution

i)
$$P\left(\left(\frac{3}{8},\frac{4}{8}\right]\cup\left(\frac{7}{8},\frac{8}{8}\right]\right)=P\left(\left(\frac{3}{8},\frac{4}{8}\right]\right)+P\left(\left(\frac{7}{8},\frac{8}{8}\right]\right)=\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$$

ii) Let
$$A=(3/8,4/8]\cup(7/8,8/8].$$
Then
$$\mathcal{F}=\{\emptyset,A,A^c,\Omega\}$$
will allow us to evaluate the probability from part (a) with just four sets.

iii) If we need to calculate the probability of any two values, consider
$$A_0=(0/8,1/8]\cup(4/8,5/8],$$
$$A_1=(1/8,2/8]\cup(5/8,6/8],$$
$$A_2=(2/8,3/8]\cup(6/8,7/8],$$
$$A_3=(3/8,4/8]\cup(7/8,8/8].$$
Then let $\mathcal{F}$ consist of the unions and complements of these four sets. This yields $2^4=16$ sets total, unlike the $\mathcal{C}$ used in part (i) which is composed of $2^8=256$ sets.

Problem 7. Probability of Union

Let P denote a probability function on sample space Ω and σ-algebra $\mathcal{F}$. Show that
$$P\left(\bigcup_{i=1}^{\infty}A_i\right)\leq\sum_{i=1}^{\infty}P(A_i).$$

Problem 7 Solution

First we will show the following:
$$P(A\cup B)\leq P(A)+P(B).$$
which can be seen since
$$P(A\cup B)=P(A\cup B)+P(A\setminus B)+P(B\setminus A)$$
$$=P(A)+P(B)-P(A\cup B)\leq P(A)+P(B).$$
Then we will use induction. Assume that
$$P\left(\bigcup_{i=1}^{n}A_i\right)\leq\sum_{i=1}^{n}P(A_i).$$

Then

$$P\left(\bigcup_{i=1}^{n+1} A_i\right) = P\left(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\right) \leq \sum_{i=1}^{n} P(A_i) + P(A_{n+1}) = \sum_{i=1}^{n+1} P(A_i).$$

where the inequality follows by setting A in the above relationship to the sum, and B to A_{n+1}. The statement follows by induction on n and taking the limit as $n \to \infty$.

Problem 8. Some Results on $\sigma-$algebras

i) Let $\mathcal{F}$ denote the collection of all countable subsets of $\Omega = \mathbb{R}$, and their complements. Show that

  a) $\mathcal{F}$ is a σ-algebra.

  b) If $P : \mathcal{F} \to [0, 1]$ is such that $P(A) = 0$ if A is countable, then $(\Omega, \mathcal{F}, P)$ forms a valid probability space.

ii) Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset$ ... denote an increasing series of σ-algebras. Show by example that $\bigcup_i \mathcal{F}_i$ is not necessary a σ-algebra.

Problem 8 Solution

i) (a) The first property holds because $\Omega^c = \emptyset$ is countable. The second property holds by construction. To show the third property, let $\{A_i\}_{i=1}^{\infty}$ denote a collection of sets in $\mathcal{F}$. Write the infinite union as

$$\left(\bigcup_{i: A_i \text{ uncountable}} A_i\right) \cup \left(\bigcup_{i: A_i \text{ countable}} A_i\right) \equiv A \cup B.$$

where we will denote the first union as just A and the second union as B for simplicity.

Since the countable union of countable sets is countable, then B is in $\mathcal{F}$. A is in $\mathcal{F}$ since its complement is countable, as

$$\left(\bigcup_i A_i\right)^c = \left(\bigcap_i A_i^c\right)$$

is the intersection of countable sets. Then we can see that

$$A \cup B = (\Omega \setminus (A^c)) \cup B = \Omega \setminus ((A^c) \setminus B).$$

Since $A^c \setminus B$ is countable, then it follows that $A \cup B$ is countable, so $\bigcup_{n=1}^{\infty} A_i \in \mathcal{F}$.

(b) Since $\emptyset$ is countable and the complement of Ω is countable, then $P(\emptyset) = 0$ and $P(\Omega) = 1$. Consider a *disjoint* collection $\{A_i\}_i$. Then it can be seen that at most one of these is uncountable (since, if A and B were disjoint and both uncountable, then B would be in A^c

which violates the construction). Say there is one uncountable set that we identify as $i = 1$. Then $\bigcup_i A_i$ is uncountable, so

$$1 = P\left(\bigcup_i A_i\right) = P(A_1) + \sum_{n=2}^{\infty} P(A_i) = 1 + 0 = 0.$$

If all of the sets are countable, then both sides will just be equal to zero.

ii) For one example, consider $\mathcal{F}_n$ as the σ-algebra obtained by taking unions/complements of half-open sub-intervals of $(0, 1]$ of length $\frac{1}{2^n}$, i.e. $\left(\frac{i}{2^n}, \frac{i+1}{2^n}\right]$. Then, for example, $\{1/2\} \notin \bigcup_{n=1}^{\infty} \mathcal{F}_i$ because $\{1/2\} \notin \mathcal{F}_n$ for any n, however $\bigcap_{n=1}^{\infty}(1/2 - 1/2^n, 1/2] = 1/2$ implies that $\{1/2\}$ would need to be in $\bigcup_{n=1}^{\infty} \mathcal{F}_i$ in order for it to be a σ-algebra.

Another example would be to let $\mathcal{F}_n$ be equal to the set of all subsets of $\{1, ..., n\}$ and their complements in $\mathbb{N}$. Then $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ will include all of the singleton sets $\{i\}$, so $\bigcup_{i:\text{ i is even}}\{i\}$ will be the set of all even natural numbers. However this set is not in any $\mathcal{F}_n$, so cannot be in $\bigcup_{n=1}^{\infty} \mathcal{F}_n$.