Stat 513

Fall 2025

Final Problem Set

Due Friday, December 19 at 23:59

Note: You can use any available resources for completing these problems, however you will benefit more by solving each question without looking (or using AI to generate) the solutions.

Topic 1. Probability, Set Theory, and Real Analysis Preliminaries

Consider a sequence of functions $\{f_n\}_{n=1}^{\infty}$. Such a sequence is said to converge uniformly to a function f on a set E if for all $\epsilon > 0$, there exists an N such that for all $n > N$,

$$|f_n(x) - f(x)| < \epsilon \text{ for all } x \in E.$$

a) The sequence $\{f_n\}$ converges pointwise to f on E if $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in E$. Demonstrate that uniform convergence is a stronger condition than pointwise convergence (i.e., uniform convergence implies pointwise convergence, but not the other way around).

b) Show that the following definition of uniform convergence is equivalent to the one given above:

$\forall\, \epsilon > 0$, there exists an N such that $m \geq N$ and $n \geq N$ implies that $|f_n(x) - f_m(x)| \leq \epsilon$ for all $x \in E$.

Topic 2. Random Variables and Distributions

Let X and Y denote two continuous random variables with joint probability density function given by

$$f_{X,Y}(x,y) = xe^{-y/\beta}/\beta^3 \text{ for } 0 < x < y$$

and $f_{X,Y}(x,y) = 0$ otherwise, where $\beta > 0$.

a) Are X and Y independent? Why or why not?

b) What is the marginal pdf of X?

c) Let $U = X/Y$. Derive the joint distribution of Y and U (this will involve Topic 7 concepts, i.e. Jacobians).

d) Are Y and U independent? Why or why not?

e) What are the marginal pdfs of Y and U?

Topic 3. Integration and Expectation

The concept of *ranks* is relates to the concept of order statistics, where if $X_1, \ldots, X_n$ denotes a sequence of i.i.d. random variables and $X_{(1)}, \ldots, X_{(n)}$ denote the corresponding order statistics, then for each $1 \leq i \leq n$, the discrete random variable R_i gives the position of the of variable X_i in the ordering; in other words, R_i is such that $X_i = X_{(R_i)}$. Find the expectation of the following quantities:

a)
$$\sum_{i=1}^{n} \sum_{j=1}^{n} R_i X_j.$$

b)
$$\sum_{i=1}^{n} R_i X_{(i)}.$$

Topic 4. Conditional Distributions and Conditional Expectation

Let X denote a random variable distributed as $U(0,1)$, and let Y denote a random variable whose conditional distribution given $X = x$ is $N(x, x^2)$.

i) Derive the values of $\mathbb{E}[X]$, $\mathbb{V}[X]$, and (X, Y).

ii) Derive the joint distribution of Y/X and X. Are Y/X and X independent?

iii) Derive the marginal distribution of the ratio Y/X, as well as the expectation $E[Y/X]$.

Topic 5. Types of Convergence of Random Variables

Let $X_1, X_2, \ldots \overset{iid}{\sim} \text{Beta}(\alpha, \beta)$. Define
$$Y_i = \frac{\alpha(X_i - 1) + \beta X_i}{(\alpha + \beta)}$$
and consider the random variable $Z_n = \sum_{i=1}^{n} Y_i / \sqrt{n}$. Find the distribution of a random variable Z such that $Z_n \xrightarrow{\mathcal{D}} Z$.

Topic 6. Moment Generating Functions, Characteristic Functions, and Cumulant Generating Functions

Show using characteristic functions that if $X \sim N(0, \sigma^2)$, then

$$\mathbb{E}[\cos(X)] = e^{-\sigma^2/2}.$$

Topic 7. Parametric Families of Distributions and Transformations of Random Variables

Let $U_1, U_2 \overset{iid}{\sim}$ Uniform$(0, 1)$, and let

$$X_1 = \sqrt{-2 \log U_1} \cos(2\pi U_2)$$
$$X_2 = \sqrt{-2 \log U_1} \sin(2\pi U_2).$$

Derive the joint density of (X_1, X_2).

Bonus Problem. Normal Distribution Theory.

Let X denote a standard normal random variable and let Y denote an independent Bernoulli(1/2) random variable. Define

$$Z = (2Y - 1)X.$$

a) Show that Z has a standard normal distribution.

b) Are X and Z independent? Why or why not?

c) Does the vector (X, Z) have a multivariate (bivariate) normal distribution? Why or why not?