Stat 513
Assignment 1
Topic 1: Probability and Set Theory
Fall 2025

Total: 50 points

Due Sunday, September 7 at 23:59

1 Set Theory

Problem 1. Countability..

The *algebraic numbers* are defined as the set of roots of polynomials with integer coefficients. Formally,

$$A = \left\{ x : \exists N, a_0, a_1, ...a_N \in \mathbb{Z} \text{ s.t. } \sum_{i=0}^{N} a_i x^i = 0 \right\}.$$

Is A countable or uncountable? Show your answer by either demonstrating the existence of a bijection, or showing that no such bijection could exist.

Problem 2. Countability and Density of Sets

As in class, let $b_i(x)$ denote the i-th binary digit of $x \in (0,1)$. The set of normal numbers between zero and one is defined as the following set:

$$A = \left\{ x \in (0,1) \middle| \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} b_i(x) = \frac{1}{2} \right\}.$$

i) Show that A is dense in $(0,1)$. *Hint:* For a given ϵ, look at the first n digits of the binary expansion for an appropriate value of n.

ii) Show that the complement of A is also dense in $(0,1)$.

Problem 3. Lim-sup and lim-inf of sets.

Consider a countable sequence of sets $A_1, A_2, \ldots$. The lim-sup and lim-inf of this sequence are defined as follows:

$$\liminf A_i = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$$

and

$$\limsup A_i = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i.$$

i) Let $A_i = \left[0, \frac{1}{i}\right]$ if i is odd and $A_i = [0,1]$ if i is even. What are $\limsup A_i$ and $\liminf A_i$?

ii) Show that if $A_i \subset A_{i+1}$ for all i, then the liminf and limsup of $\{A_i\}_{i=1}^{\infty}$ are equal to each other and to the (infinite) union. Show the analogous result if $A_i \supset A_{i+1}$ with respect to the infinite intersection.

iii) Show that

$$(\liminf A_i)^c = \limsup A_i^c$$

and

$$(\limsup A_i)^c = \liminf A_i^c.$$

Problem 4. The Cantor Set

Consider the set of sequences of elements of $\{0, 1, 2\}$:

$$\mathcal{T} = \left\{ \{x_i\}_{i=1}^{\infty} \middle| x \in \{0, 1, 2\} \right\}.$$

Similar to the set of binary sequences, we can define $t_i(x)$ as the i-th *ternary* digit of $x \in (0, 1)$, and establish a (psuedo) bijection with the unit interval:

$$f(\{x_i\}) = \sum_{i=1}^{\infty} \frac{t_i(x)}{3^i}.$$

(Note that this is not a true bijection as written because we would need to establish a condition for equivalent expansions similar to what we did for the binary digits, since $.022\ldots$ and $.100\ldots$ are both equal to $1/3$; however we will ignore this complication as justified by Problem 5(i)).

Given this definition, the *Cantor set* can be defined as

$$\mathcal{C} = \{x \in (0, 1) \big| t_i(x) \neq 1\}.$$

i) Show that the Cantor set as defined above is equivalent to defining collections of sets C_n for all $n \in \mathbb{N}$ through the following iterative process:

   a) Initialization: Let $C_0 = \{(0, 1)\}$.

   b) Construct C_{n+1} from C_n by removing the middle third of each of the intervals of C_n, i.e.

   $$C_{n+1} = \left\{ \left[a, a + \frac{b-a}{3}\right], \left[a + \frac{2(b-a)}{3}, b\right] \middle| \forall [a, b] \in C_n \right\}.$$

   and taking the infinite intersection of the union of each of the C_n:

   $$\mathcal{C} = \bigcap_{n=1}^{\infty} \bigcup_{A \in C_n} A.$$

ii) Show that the Cantor set is closed (i.e., contains all of its limit points).

iii) A set S is *nowhere dense* in $\mathcal{X}$ if for all open subsets $E \subset \mathcal{X}$, S is not dense in E. Show that the Cantor set is nowhere dense in $(0,1)$.

iv) Using the uniform probability space, show that $P(\mathcal{C}) = 0$ by showing that $P(\mathcal{C}) < \epsilon$ for all $\epsilon > 0$.

2 Basic Probability

Problem 5. Binary Sequences

i) Show that, when represented in base-2, .1000... = .0111... = $\frac{1}{2}$.

ii) Taking $(\Omega, \mathcal{F}, P)$ to be $\Omega = (0,1)$, $\mathcal{F}$ as the Borel sets, and P as the uniform probability measure, show that

$$P\left(\left\{x \middle| \exists N \text{ s.t. } b_j(x) = 0 \; \forall j \geq N\right\}\right) = 0.$$

(In other words, the probability that a given binary sequence ends in all zeros is zero). This allows us to use our bijection between the "non-terminating" binary sequences and the interval $(0,1)$ without any loss of generality.

Problem 6. Infinite Sequences of Coin Flips

For the following parts, consider the event space of infinite sequences of zero-one coin flips:

$$\Omega = \left\{ \{x_i\}_{i=1}^{\infty} \middle| x_i \in \{0,1\} \right\}.$$

i) Using the $\sigma-$algebra generated by evenly sized intervals of width 1/8 (i.e., $\mathcal{F}$ composed of the sets $A_i = (i/8, (i+1)/8)$ along with union and complements) derive the probability of the second and third coin flips being heads.

ii) What is the smallest σ-algebra (i.e., composed of the fewest number of sets) that will allow you to evaluate the probability of the second and third coin flips being heads?

iii) What is the smallest σ-algebra (i.e., composed of the fewest number of sets) that will allow you to evaluate the probabilities that the second and third coin flips take *any* value? (i.e, (H,H), (T,T), (H,T), (T,T)).

Problem 7. Probability of Union

Let P denote a probability function on sample space Ω and σ-algebra $\mathcal{F}$. Show that

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

Problem 8. Some Results on $\sigma-$algebras

i) Let $\mathcal{F}$ denote the collection of all countable subsets of $\Omega = \mathbb{R}$, and their complements. Show that

  a) $\mathcal{F}$ is a σ-algebra.

  b) If $P : \mathcal{F} \to [0, 1]$ is such that $P(A) = 0$ if A is countable, then $(\Omega, \mathcal{F}, P)$ forms a valid probability space.

ii) Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset ...$ denote an increasing series of σ-algebras. Show by example that $\bigcup_i \mathcal{F}_i$ is not necessary a σ-algebra.

Bonus Problem. Infinite Monkeys.

Let $\Omega = \left\{ \{x_i\}_{i=1}^{\infty} \middle| x_i \in \{a, b, ..., z\} \right\}$ denote the collection of infinite sequences of latin letters. Note that we can define a "uniform" probability space on Ω through a bijection between Ω and the interval $(0, 1)$ (using numbers represented in base 26).

Let $S = (x_1, ..., x_n)$ denote a fixed sequence of n letters. Show that for any such sequence,

$$P\left(\left\{ \{x_i\} \in \Omega \middle| S \text{ is a sub-sequence of } \{x_i\} \right\}\right) = 1.$$